Physica D (2007), doi:10.1016/j.physd.2007.09.020

Variational formulation of the motion of an ideal fluid on the basis of gauge principle

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On the basis of gauge principle in the field theory, a new variational formulation is presented for flows of an ideal fluid. The fluid is defined thermodynamically by mass density and entropy density, and its flow fields are characterized by symmetries of translation and rotation. A structure of rotation symmetry is equipped with a Lagrangian Λ_A including vorticity, in addition to Lagrangians of translation symmetry. From the action principle, Euler's equation of motion is derived. In addition, the equations of continuity and entropy are derived from the variations. Equations of conserved currents are deduced as the Noether theorem in the space of Lagrangian coordinate $\boldsymbol{a}$. It is shown that, with the translation symmetry alone, there is freedom in the transformation between the Lagrangian $\boldsymbol{a}$-space and Eulerian $\boldsymbol{x}$-space. The Lagrangian Λ_A provides non-trivial topology of vorticity field and yields a source term of the helicity. The vorticity equation is derived as an equation of the gauge field. Present formulation provides a basis on which the transformation between the $\boldsymbol{a}$ space and the $\boldsymbol{x}$ space is determined uniquely.

PACS numbers: 47.10.-g, 03.50.-z, 47.15.ki, 47.90.+a
Keywords: Variational formulation, Gauge principle, Euler's equation, Helicity, Chern-Simons term

I. INTRODUCTION

In the historical paper '*General laws of the motion of fluids*' [1], Leonhard Euler verified that his equation of motion can describe rotational flows. The same theme is investigated in this paper under modern view. *Fluid mechanics* is understood to be a field theory in Newtonian mechanics that has Galilean symmetry. It is covariant under transformations of the Galilei group. The gauge principle ([19], [20], [21]) requires a physical system under investigation to have a *symmetry*, *i.e.* a gauge invariance with respect to a certain group of transformations. Following this principle, the gauge symmetry of flow fields is studied in [2] and [3] with respect to both translational and rotational transformations. The formulation started from a Galilei-invariant Lagrangian of a system of *point masses* which is known to have *global* gauge symmetries with respect to both translation and rotation [4]. It was then extended to flows of a fluid, a continuous material characterized with mass density and entropy density. In addition to the global symmetry, *local* gauge invariance of a Lagrangian is required for such a continuous field. Symmetries imply conservation laws. Equations of conserved currents are deduced as the Noether theorem.

Thus, the convective derivative of fluid mechanics, *i.e.* the Lagrange derivative, is identified as the *covariant derivative*, which is a building block in the framework of gauge theory. Based on this, appropriate Lagrangians are defined for motion of an ideal fluid. Euler's equation of motion is derived from the action principle. In most traditional formulations, the continuity equation and entropy equation are given as constraints for the variations, while in this new formulation those equations were derived from the action principle. In the previous study ([5], [6]) of rotational symmetry of the velocity field $\boldsymbol{v}(\boldsymbol{x})$, it is found that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the gauge field associated with the rotational symmetry of velocity.

A new structure of the rotational symmetry was given in [3] by the following Lagrangian:

$$\Lambda_A = -\int_M \langle \mathcal{L}_W \boldsymbol{A},\, \boldsymbol{\omega} \rangle \, \mathrm{d}^3\boldsymbol{x},$$

where $\boldsymbol{A}$ is a vector potential and $\mathcal{L}_W \boldsymbol{A} = \partial_t \boldsymbol{A} + v^k \partial_k \boldsymbol{A} + A_k \nabla v^k$. This is derived from a representation characteristic of a *topological* term known in the gauge theory. This yields non-vanishing rotational component of the velocity field, and provides a source term of helicity. This is closely related to the Chern-Simons term, describing non-trivial topology of vorticity field, *i.e.* mutual linking of vorticity lines. The vorticity equation is derived as an equation for the gauge field.

With regard to the variational formulation of fluid flows, the papers [7] and [8] are among the earliest to have influenced current formulations. Their variations are carried out in two ways: *i.e.* a Lagrangian approach and an Eulerian approach. In both approaches, the equation of continuity and the condition of isentropy are added as constraint conditions on the variations by means of Lagrange multipliers. The Lagrangian approach is also taken by [9]. In this relativistic formulation those equation are derived from the equations of current conservation. Several action principles to describe relativistic fluid dynamics have appeared in the past (see [9, §4.2] for some list of them).

In the Lagrangian approach, the Euler-Lagrange equation results in an equation equivalent to Euler's equation

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of motion in which the acceleration term is represented as the second time derivative of position coordinates of the Lagrangian representation. In this formulation, however, there is a certain degree of freedom in the relation between the Lagrangian particle coordinates and Eulerian space coordinates. Namely, the relation between them is determined only up to an unknown rotation. In the second approach referred to as the *Eulerian* description, the action principle of an ideal fluid results in potential flows with vanishing helicity, if the fluid is *homentropic* [2]. However, as noted in the beginning, it should be possible to have rotational flows even in such a homentropic fluid. Gauge theory for fluid flows provides a crucial key to resolve these issues. It was also shown in [2] that a general solution in the translational symmetry alone is equivalent to the classical Clebsch solution [10]. A new formulation on the basis of the Clebsch parametrization is carried out in [11] and [12] aiming at its extension to supersymmetric and non-Abelian fluid mechanics.

It is interesting to note the gauge invariances known in the theory of electromagnetism and fluid flows. There is an invariance under a gauge transformation of electromagnetic potentials consisting of a scalar potential ϕ and a vector potential $\boldsymbol{A}$. An analogous invariance is pointed out in [2] for a gauge transformation of a velocity potential ϕ of irrotational flows of an ideal fluid, where the velocity is represented as $\boldsymbol{v} = \nabla\phi$. It is shown in [7] (cited in [3]) that gauge invariance is not restricted to the potential flows, but also there is known an invariance in the rotational flow of *Clebsch representation*.

II. EQUATIONS IN $\boldsymbol{a}$-SPACE

A. Lagrangian

Let us consider a variational formulation with a Lagrangian represented with the particle coordinate $\boldsymbol{a} = (a^1, a^2, a^3) = (a, b, c)$ (*i.e. Lagrangian* coordinates). Independent variables are denoted with a^μ where μ or greek letter suffix take $= 0, 1, 2, 3$ with a^0 the time variable written also as τ $(= t)$: $a^\mu = (\tau, a^1, a^2, a^3)$. Corresponding physical space coordinate $\boldsymbol{x} = (x, y, z)$ (*Eulerian* coordinates) are written also as $x^\mu = (t, x^1, x^2, x^3)$. The letter τ is used (instead of t) in combination with the particle coordinates a^k. Physical space position of a particle $\boldsymbol{a}$ is expressed by $X^k(a^\mu) = X^k(\tau, \boldsymbol{a})$, or $X^k = (X, Y, Z)$. Its velocity is given by $v^k = \partial_\tau X^k$, also written as X^k_τ.

The Lagrangian coordinates (a, b, c) are defined such that an infinitsimal three-element $\mathrm{d}^3\boldsymbol{a} = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$ denotes a mass element $\mathrm{d}m$ of an infinitesimal volume $\mathrm{d}^3\boldsymbol{x} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ of the $\boldsymbol{x}$-space. The mass elelment $\mathrm{d}m$ should be invariant during the motion:

$$\partial_\tau(\mathrm{d}m) \equiv \partial_\tau(\mathrm{d}^3\boldsymbol{a}) = 0\,. \tag{1}$$

The mass-density ρ is defined by the equation $\mathrm{d}^3\boldsymbol{a} = \rho\,\mathrm{d}^3\boldsymbol{x}$. With using a Jacobian determinant J of the transformation $X^k = X^k(a^l)$ from $\boldsymbol{a}$-space to $\boldsymbol{X}$-space $(k,\, l = 1, 2, 3)$, we have

$$\rho = \frac{1}{J}, \qquad J = \frac{\partial(X^1, X^2, X^3)}{\partial(a^1, a^2, a^3)} = \frac{\partial(X, Y, Z)}{\partial(a, b, c)}\,. \tag{2}$$

In an ideal fluid, there is no dissipation of kinetic energy into heat, by definition. According to thermodynamics for the entropy s (per unit mass) and temperature T, we have $T\delta s = 0$ if there is no heat production. Namely the entropy s does not depend on τ. Then, the change of internal energy ϵ (per unit mass) is related to the density change $\delta\rho$ alone by

$$\delta\epsilon = (\delta\epsilon)_s = \frac{p}{\rho^2}\,\delta\rho, \quad \Big(\frac{\partial\epsilon}{\partial\rho}\Big)_s = \frac{p}{\rho^2}, \quad \delta h = \frac{1}{\rho}\delta p, \tag{3}$$

where p is the fluid pressure, and $h = \epsilon + p/\rho$ the enthalpy, and $(\,\cdot\,)_s$ denotes s being fixed. However, the entropy s may not be uniform and may depend on $\boldsymbol{a}$ by initial condition. Hence, $s = s(\boldsymbol{a})$, or equivqalently,

$$\partial_\tau\, s = 0\,. \tag{4}$$

Total Lagrangian is defined by

$$\Lambda_{\mathrm{T}} = \int_{M_a} \tfrac{1}{2}\, X^k_\tau\, X^k_\tau\, \mathrm{d}^3\boldsymbol{a} - \int_{M_a} \epsilon(\rho, s)\, \mathrm{d}^3\boldsymbol{a}\,, \tag{5}$$

[3], where M_a is a space of fluid under investigation, and $X^k_\tau = X^k_0 = v^k$ is the velocity. The internal energy $\epsilon(\rho, s)$ of the second term depends on ρ (which in turn depends on $X^k_l = \partial X^k/\partial a^l$ by (2)) and the entropy $s(\boldsymbol{a})$.

An action I is defined by the integral: $I = \int \Lambda_{\mathrm{T}}\mathrm{d}\tau$:

$$I \;=\; \int L(X^k_\mu)\; \mathrm{d}^4 a, \qquad \mathrm{d}^4 a = \mathrm{d}\tau\, \mathrm{d}^3\boldsymbol{a}, \tag{6}$$

$$L(X^k_\mu) = \tfrac{1}{2}\, X^k_0\, X^k_0 - \epsilon(X^k_l, a^k). \tag{7}$$

B. Noether's theorem

Euler-Lagrange equation associated with the Lagrangian (7) is given by

$$\frac{\partial}{\partial a^\mu}\Big(\frac{\partial L}{\partial X^k_\mu}\Big) - \frac{\partial L}{\partial X^k} = \partial_\mu\Big(\frac{\partial L}{\partial X^k_\mu}\Big) - \frac{\partial L}{\partial X^k} = 0. \tag{8}$$

Energy-momentum tensor T^ν_μ is defined by

$$T^\nu_\mu \equiv X^k_\mu \Big(\frac{\partial L}{\partial X^k_\nu}\Big) - L\,\delta^\nu_\mu\,, \tag{9}$$

[7], where $k = 1, 2, 3$. As long as (8) is satisfied together with an assumption of τ-independence of L (*i.e.* $\partial_\tau L = 0$), it can be verified [3] that we have a conservation equation $\partial_\nu T^\nu_\mu = 0$ (where $\partial_\mu = \partial/\partial a^\mu$). This is the Noether theorem ([13], [19]).

For $\mu \neq 0$ $(x^\mu = \alpha)$, the conservation law $\partial_\nu T^\nu_\mu = 0$ reduces to the momentum equaitons:

$$\partial_\tau V_\alpha + \partial_\alpha\, F = 0 \quad (V_\alpha \equiv X_\alpha X_\tau + Y_\alpha Y_\tau + Z_\alpha Z_\tau), \tag{10}$$

[7], where $F = -\frac{1}{2}v^2 + h$. Two other equations are obtained with α replaced by cyclic permutaion of (a, b, c). Integrating this with respect to τ between 0 and t, we find the Weber's transformation [14, Art.15]:

$$V_\alpha(\tau) \equiv X_\alpha X_\tau + Y_\alpha Y_\tau + Z_\alpha Z_\tau = V_\alpha(0) - \partial_\alpha \chi, \quad (11)$$
$$\chi = \int_0^t F\,\mathrm{d}\tau = \int_0^t (-\tfrac{1}{2}v^2 + h)\mathrm{d}\tau.$$

The V_α of (10) is a transformed velocity in the $\boldsymbol{a}$-space (Sec.V A). Its time evolution is given by (11) for a given initial values of $V_\alpha(0, \boldsymbol{a})$ and $h(0, \boldsymbol{a})$ at $\boldsymbol{a} = \boldsymbol{x}$.

With $\mu = 0$, we have the energy equation:

$$\partial_\tau H \;+\; \partial_a\Big[p\frac{\partial(X,Y,Z)}{\partial(\tau,b,c)}\Big] + \partial_b\Big[p\frac{\partial(X,Y,Z)}{\partial(a,\tau,c)}\Big] \\ +\; \partial_c\Big[p\frac{\partial(X,Y,Z)}{\partial(a,b,\tau)}\Big] = 0. \quad (12)$$

where $H = \frac{1}{2}v^2 + \epsilon$. The equation (10) reduces to the equation for the acceleration $\mathcal{A}_\alpha(\tau, \boldsymbol{a})$:

$$\mathcal{A}_\alpha \equiv X_\alpha X_{\tau\tau} + Y_\alpha Y_{\tau\tau} + Z_\alpha Z_{\tau\tau} = -\frac{1}{\rho}\,\partial_\alpha p, \quad (13)$$

which is known as the Lagrangian form of equation of motion [14, Art.13]. This can be transformed to

$$X_{\tau\tau} = -\frac{1}{\rho}\,\partial_x\, p, \qquad \partial_x p = \frac{\partial \alpha}{\partial x}\,\frac{\partial p}{\partial \alpha} \quad (14)$$

[3]. Since $X_{\tau\tau}$ is the x-accceleration of the particle $\boldsymbol{a}$, this is the form equivalent to the x-component of Euler's equation of motion (25). The y and z components can be obtained analogously.

C. Arbitrariness in the transformation

There is an arbitrariness in the transformation from the $\boldsymbol{a}$-space to the $\boldsymbol{x}$-space with respect to the equation (13). Its middle-side expression is a form of scalar product of two vectors in the $\boldsymbol{x}$-space: the particle accelerataion $(X_{\tau\tau}, Y_{\tau\tau}, Z_{\tau\tau})$ and the direction vector $(X_\alpha, Y_\alpha, Z_\alpha)$ of the α-axis in the $\boldsymbol{a}$-space.

Putting it in a different way, the equation (13) is invariant with respect to orthogonal rotational transformations of a displacement vector $\Delta\boldsymbol{X} = (\Delta X, \Delta Y, \Delta Z)$ of a particle in the $\boldsymbol{x}$-space. In fact, suppose that a vector $\underline{\Delta\boldsymbol{X}}$ satisfies the equation (13). Then, another vector $\overline{\Delta\boldsymbol{X}} = R\,\Delta\boldsymbol{X}$ obtained by an orthogonal transformation R satisfies the same equation, since any orthogonal matrix satisfies $RR^T = I$ (unit mtrix) where R^T denotes the transposed matrix of R. So that the vector $\Delta\boldsymbol{X}$ is not uniquely determined. The same freedom can be said to the velocity $V_\alpha(\tau, \boldsymbol{a})$ of (11) as well.

These imply that a certain machinery must be equipped in order to fix this arbitrariness within the framework of rotational symmetry. This will be considered later. Note that the density ρ is not changed by the orthogonal transformation.

III. EQUATIONS IN $\boldsymbol{x}$-SPACE

A. Action in Eulerian representation

Eulerian description is represented by the independent variables (t, x, y, z). Local gauge symmetries of fluid flows are investigated in detail in [2], [3]. The time derivative ∂_τ is equivalent to the convective derivative D_t:

$$\partial_\tau = \mathrm{D}_t, \quad \mathrm{D}_t \equiv \partial_t + u\partial_x + v\partial_y + w\partial_z = \partial_t + \boldsymbol{v}\cdot\nabla\,. \quad (15)$$

The operator D_t is verified to be gauge-invariant. The velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is defined by the particle velocity:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \partial_\tau \boldsymbol{X} = \mathrm{D}_t \boldsymbol{x}. \quad (16)$$

The acceleration field $\mathcal{A}(\boldsymbol{x}, t)$ is also defined by

$$\mathcal{A}(\boldsymbol{x}, t) = \partial_\tau{}^2 \boldsymbol{X} = \mathrm{D}_t \boldsymbol{v} = (\partial_t + v^k\partial_k)\boldsymbol{v}. \quad (17)$$

As noted previously, the mass $\mathrm{d}^3\boldsymbol{a}(\boldsymbol{a})$ and the entropy $s = s(\boldsymbol{a})$ satisfy (1) and (4). In view of these properties, we can define the following two Lagrangians:

$$L_\phi = -\int_M \partial_\tau\phi\,\mathrm{d}^3\boldsymbol{a}, \qquad L_\psi = -\int_M s\,\partial_\tau\psi\,\mathrm{d}^3\boldsymbol{a}, \quad (18)$$

where $\phi(\boldsymbol{a}, \tau)$ and $\psi(\boldsymbol{a}, \tau)$ are scalar fields associated with mass and entropy, respectively. By adding L_ϕ and L_ψ to Λ_T of (5), the total Lagrangian is given by

$$\Lambda_\mathrm{T}^* = \Lambda_\mathrm{T} - \int \partial_\tau\phi\;\mathrm{d}^3\boldsymbol{a} - \int s\,\partial_\tau\psi\;\mathrm{d}^3\boldsymbol{a}. \quad (19)$$

The action is defined by $I = \int_{\tau_1}^{\tau_2} \Lambda_\mathrm{T}^* \mathrm{d}\tau$, where the integral $I_\phi = \int \mathrm{d}\tau \int \partial_\tau\phi\,\mathrm{d}^3\boldsymbol{a}$ can be integrated with respect to τ and expressed as $\int[\phi]\mathrm{d}^3\boldsymbol{a}$, where $[\phi] = \phi|_{\tau_2} - \phi|_{\tau_1}$ is the difference of ϕ at the end times τ_2 and τ_1 and hence independent of $\tau \in (\tau_1, \tau_2)$. Likewise, the last integral can be expressed as $I_\psi = \int[\psi]s\,\mathrm{d}^3\boldsymbol{a}$, because s is independent of τ. This means that the gauge potentials ϕ and ψ do not appear in the equation of motion obtained through variations of the action I for $\tau \in (\tau_1, \tau_2)$.

However, it becomes soon clear that these are non-trivial in the expressions of the $\boldsymbol{x}$-space, because they are rewritten as $L_\phi = -\int_M \rho\,\mathrm{D}_t\phi\,\mathrm{d}^3\boldsymbol{x}$, and $L_\psi = -\int_M \rho s\,\mathrm{D}_t\psi\,\mathrm{d}^3\boldsymbol{x}$ by using the relations $\mathrm{d}^3\boldsymbol{a} = \rho\,\mathrm{d}^3\boldsymbol{x}$ and $\partial_\tau = \mathrm{D}_t$.

In the $\boldsymbol{x}$-space, the total Lagrangian can be written as $\Lambda_\mathrm{T}^* = \int_M \mathcal{L}(\boldsymbol{v}, \rho, s, \phi, \psi)\;\mathrm{d}^3\boldsymbol{x}$, where

$$\mathcal{L} \equiv \tfrac{1}{2}\rho\, v^k v^k - \rho\epsilon(\rho, s) - \rho\mathrm{D}_t\phi - \rho s\mathrm{D}_t\psi \quad (20)$$

[22]. This is proposed as a *possible* form of Lagrangian in the $\boldsymbol{x}$-space (but an additional term will be added later). The action is defined by $I = \int \mathcal{L}(\boldsymbol{v}, \rho, s, \phi, \psi)\;\mathrm{d}^4x$, where $\mathrm{d}^4x = \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$. However, the action principle results in the potential flow represented by $\boldsymbol{v} = \mathrm{grad}(\phi + s_0\psi)$ when the fluid has a uniform entropy s_0 (see [2]).

B. Outcomes of variations

We require invariance of the action I with respect to variantions. First, consider the following infinitesimal transformation: $\boldsymbol{x}'(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x},t)$. The volume element $\mathrm{d}^3\boldsymbol{x}$ is changed to $\mathrm{d}^3\boldsymbol{x}' = (1+\partial_k\xi^k)\mathrm{d}^3\boldsymbol{x}$, up to the first order terms. Hence the variation of volume is given by $\Delta(\mathrm{d}^3\boldsymbol{x}) = \partial_k\xi^k\,\mathrm{d}^3\boldsymbol{x}$, while the variations of density, velocity and entropy are $\Delta\rho = -\rho\,\partial_k\xi^k$, $\Delta\boldsymbol{v} = \mathrm{D}_t\boldsymbol{\xi}$, and $\Delta s = 0$. Under these together with (1) and (4) (with keeping ϕ and ψ fixed), the variation of I is given by

$$\Delta I = \int \mathrm{d}^4x \left[\frac{\partial L}{\partial \boldsymbol{v}}\Delta\boldsymbol{v} + \frac{\partial L}{\partial \rho}\,\Delta\rho + \frac{\partial L}{\partial s}\,\Delta s + L\,\partial_k\xi^k \right].$$

This is required to vanish for arbitrary variation of ξ^k, which results in the Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial v^k}\Big) + \frac{\partial}{\partial x^l}\Big(v^l\,\frac{\partial L}{\partial v^k}\Big) + \frac{\partial}{\partial x^k}\Big(L - \rho\,\frac{\partial L}{\partial \rho}\Big) = 0. \quad (21)$$

Similarly, invariance of I with respect to arbitrary variations of ϕ and ψ (denoted by $\Delta\phi$ and $\Delta\psi$) leads to

$$\Delta\phi\ :\ \ \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\ \ (\textit{continuity equation}), \quad (22)$$
$$\Delta\psi\ :\ \ \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0\,. \quad (23)$$

C. Noether's theorem in Eulerian representation

Associated with (21), one can define the momentum density m_k and momentum-flux tensor M_k^l by

$$m_k = \frac{\partial L}{\partial v^k}, \qquad M_k^l = v^l\,\frac{\partial L}{\partial v^k} + \big(L - \rho\,\frac{\partial L}{\partial \rho}\big)\,\delta_k^l\,. \quad (24)$$

From (7), we obtain $m_k = \rho v_k$ and $M_k^l = \rho v_k v^l + p\,\delta_k^l$, where $v_k = v^k$ in the present Eucledian space. The equation (21) can be written in the form of momentum conservation, $\partial_t\big(\rho v^k\big) + \partial_l\big(\rho v^l v^k\big) + \partial_k p = 0$ ($\partial_k = \partial/\partial x^k$). Using (22), this equation can be reduced to the following Euler's equation of motion:

$$\partial_t v^k + (v^l\partial_l)v^k = -\frac{1}{\rho}\,\partial_k\,p \quad (= -\partial_k\,h). \quad (25)$$

The equation (14) is equivalent to this equation.

The energy equation (12) can be transformed to the following equation of energy conservation:

$$\partial_t\left[\rho(\tfrac{1}{2}\,v^2 + \epsilon)\right] + \partial_k\left[\rho v^k\,(\tfrac{1}{2}\,v^2 + h)\right] = 0.$$

IV. ROTATION SYMMETRY

A topological structure of vorticity field is now considered with respect to the rotational symmetry. Related gauge group is the rotation group $SO(3)$. An infinitesimal rotation is described by the Lie algebra $\mathbf{so}(3)$ of three dimensions, which is non-Abelian.

From the study of the rotational gauge transformation [3], it is found that the covariant derivative ∇_t, velocity $\boldsymbol{v}$ and acceleration $\mathcal{A}$ are represented as

$$\nabla_t \ = \ \partial_t + (\boldsymbol{v}\cdot\nabla), \quad (26)$$
$$\boldsymbol{v} = \nabla_t\boldsymbol{x} \ = \ (\partial_t + (\boldsymbol{v}\cdot\nabla))\boldsymbol{x}, \quad (27)$$
$$\mathcal{A} = \nabla_t\boldsymbol{v} \ = \ \partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} \quad (28)$$
$$\nabla_t\boldsymbol{v} \ = \ \partial_t\boldsymbol{v} + \mathrm{grad}(\tfrac{1}{2}\,v^2) + \boldsymbol{\omega}\times\boldsymbol{v}. \quad (29)$$

It is verified that the last expression of $\nabla_t\boldsymbol{v} = \partial_t\boldsymbol{v} + \nabla(\frac{1}{2}v^2) + \boldsymbol{\omega}\times\boldsymbol{v}$ not only satisfies the rotational gauge-invariance, but also expresses that $\boldsymbol{\omega}$ is the gauge field of the rotational symmetry. In addition, it satisfies the covariance requirement with respect to Galilean transformation from one reference frame (t, $\boldsymbol{x}$, $\boldsymbol{v}$) to another (t_*, $\boldsymbol{x}_*$, $\boldsymbol{v}_*$) moving with a uniform relative velocity $\boldsymbol{U}$, where $t_* = t$, $\boldsymbol{x}_* = \boldsymbol{x} - \boldsymbol{U}t$ and $\boldsymbol{v}_* = \boldsymbol{v} - \boldsymbol{U}$. Namely, we have the covariance $\nabla_t\boldsymbol{v} = (\nabla_t\boldsymbol{v})_*$.

V. LAGRANGIAN ASSOCIATED WITH ROTATION SYMMETRY

Associated with the rotation symmetry, an additional Lagrangian is to be defined according to the gauge principle. It is important to observe from Sec.III A that, in the Lagrangian (19), the integrands of the last two integrals are of the form $\partial_\tau(\,\cdot\,)$. The action is defined by $I = \int\int[\Lambda_{\mathrm{T}} + \partial_\tau(\,\cdot\,)]\,\mathrm{d}\tau\mathrm{d}^3\boldsymbol{a}$. This property is regarded as the simplest representation of topology in the gauge theory ([16] $\sim$ [18], [19]). In the context of rotational flows, it is known that the helicity (or Hopf invariant, [15]) describes non-trivial topology of vorticity field, *i.e.* mutual linking of vorticity lines. This is closely related with the Chern-Simons term (without third-order term) in the gauge theory. This term lives in one dimension lower than the original four space-time (x^μ) of the action I because a topological term in the action is expressed in a form of total divergence ($\partial_\mu F^\mu$) and characterizes topologically non-trivial structures of the gauge field.

However, we learn here from the formulation of Sec.III A and look for a τ-independent field directly.

A. Lagrangian Λ_A and helicity

The τ-independent field can be found immediately from Eq. (10). Taking the curl of this equation with respect to the coordinates (a,b,c), we obtain

$$\nabla_a\times\partial_\tau\boldsymbol{V}_a = \partial_\tau(\nabla_a\times\boldsymbol{V}_a) = 0, \quad (30)$$

where $\nabla_a = (\partial_a,\partial_b,\partial_c)$. Hence, one may write as $\nabla_a\times\boldsymbol{V}_a = \boldsymbol{\Omega}_a(\boldsymbol{a})$ [7].

The vector $\boldsymbol{V}_a$ is a transformed form of the velocity $\boldsymbol{v} = (X_\tau, Y_\tau, Z_\tau) = (u,v,w)$ into the $\boldsymbol{a}$-space. This is seen on the basis of a 1-form V^1 defined by

$$V^1 \ = \ V_a\,\mathrm{d}a + V_b\,\mathrm{d}b + V_c\,\mathrm{d}c \quad (31)$$
$$\ = \ u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z. \quad (32)$$

where $V_a = ux_a + vy_a + wz_a$, $x_a = \partial X/\partial a$, $u = X_\tau$, *etc.*. Its differential $\mathrm{d}V^1$ gives a two-form $\Omega^2 = \mathrm{d}V^1$:

$$\begin{aligned}\Omega^2 &= \Omega_a \mathrm{d}b \wedge \mathrm{d}c + \Omega_b \mathrm{d}c \wedge \mathrm{d}a + \Omega_c \mathrm{d}a \wedge \mathrm{d}b \\ &= \omega_x \mathrm{d}y \wedge \mathrm{d}z + \omega_y \mathrm{d}z \wedge \mathrm{d}x + \omega_z \mathrm{d}x \wedge \mathrm{d}y, \end{aligned} \qquad (33)$$

where $(\Omega_a, \Omega_b, \Omega_c) = \boldsymbol{\Omega}_a$, and $\nabla \times \boldsymbol{v} = (\omega_x, \omega_y, \omega_z) = \boldsymbol{\omega}$ is the vorticity. Thus, it is seen that $\boldsymbol{\Omega}_a$ is the vorticity transformed to the $\boldsymbol{a}$-space. The equation (30) is transformed into the τ-derivative of the 2-form Ω^2, $\mathcal{L}_{\partial_\tau}\Omega^2 = 0$ (understood as the Lie derivative).

Next, let us introduce a gauge-potential vector $\boldsymbol{A}_a = (\overline{A}_a, \overline{A}_b, \overline{A}_c)$ in the $\boldsymbol{a}$-space, and define its 1-form A^1 by $A^1 = \overline{A}_a\, \mathrm{d}a + \overline{A}_b\, \mathrm{d}b + \overline{A}_c\, \mathrm{d}c = \overline{A}_x\, \mathrm{d}x + \overline{A}_y\, \mathrm{d}y + \overline{A}_z\, \mathrm{d}z$. Thus, it is proposed that a *possible* type of Lagrangian is

$$\Lambda_A = -\int_M \langle \partial_\tau \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} = \int_M \langle \boldsymbol{A},\, E_W[\boldsymbol{\omega}] \,\rangle\, \mathrm{d}^3\boldsymbol{x},$$

where $E_W[\boldsymbol{\omega}] \equiv \partial_t \boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{\omega}$.

New results were deduced from this Lagrangian in [3]: (i) the velocity $\boldsymbol{v}$ includes a new rotational term, (ii) the vorticity equation is derived from the variation of $\boldsymbol{A}$:

$$E_W[\boldsymbol{\omega}] = \partial_t \boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{\omega} = 0,$$

and (iii) we have non-vanishing helicity H, where

$$H = \int_V \boldsymbol{\omega}\cdot\boldsymbol{v}\; \mathrm{d}^3\boldsymbol{x} = \int_V \boldsymbol{\omega}\cdot \frac{E_W[\mathrm{curl}\boldsymbol{A}]}{\rho}\, \mathrm{d}^3\boldsymbol{x},$$

B. Uniqueness of transformation

Transformation from the Lagrangian $\boldsymbol{a}$ space to Eulerian $\boldsymbol{x}(\boldsymbol{a})$ space is determined locally by nine components of the matrix $\partial x^k/\partial a^l$. However, in the previous solution considered in Sec.II C, we had three relations (11) between $\boldsymbol{v} = (X_\tau, Y_\tau, Z_\tau)$ and (V_a, V_b, V_c), and another three relations (13) between $\mathcal{A} = (X_{\tau\tau}, Y_{\tau\tau}, Z_{\tau\tau})$ and $(\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c)$. Remaining three conditions are given by the equation (33) connecting $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ and $\boldsymbol{\Omega}_a(\boldsymbol{a}) = (\Omega_a, \Omega_b, \Omega_c)$. For example, Ω_a is determined by

$$\begin{aligned}\Omega_a &= \omega_x\, (\partial_b y\, \partial_c z - \partial_c y\, \partial_b z) + \omega_y\, (\partial_b z\, \partial_c x - \partial_c z\, \partial_b x) \\ &\quad + \omega_z\, (\partial_b x\, \partial_c y - \partial_c x\, \partial_b y). \end{aligned} \qquad (34)$$

There are three vectors (velocity, acceleration and vorticity) determined by evolution equations subject to initial conditions in each space of $\boldsymbol{x}$ and $\boldsymbol{a}$ coordinates. Transformation relations of the three vectors suffice to determine the nine matrix elemets $\partial x^k/\partial a^l$ locally. Thus, the transformation between the Lagrangian $\boldsymbol{a}$ space and Eulerian $\boldsymbol{x}(\boldsymbol{a})$ space is determined uniquely. [3]

VI. SUMMARY AND DISCUSSION

Following the scenario of the gauge principle of field theory, it is found that the variational principle of fluid motions can be reformulated successfully in terms of covariant derivative and Lagrangians defined appropriately. The present variational formulation is self-consistent and comprehensively describes flows of an ideal fluid.

In the improved formulation taking account of the rotational symmetry with additional equations of (33), the transformation relations of the three vectors (velocity, acceleration and vorticity) suffice to determine the nine matrix elements $\partial x^k/\partial a^l$ locally. Thus, the transformation between the Lagrangian $\boldsymbol{a}$ space and Eulerian $\boldsymbol{x}(\boldsymbol{a})$ space is determined uniquely.

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[22] Roman Jackiw [11] arrived at the same form, but with a different approach using Lagrannge multipliers for constraint conditions, in order to extend it to the relativistic case, *e.g.* Eq.(2.56) of [11] and (1.2.68) of [12].